Scaling behavior of transition to chaos in quasiperiodically driven dynamical systems

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A route to chaos in quasiperiodically driven dynamical systems is investigated whereby the Lyapunov exponent passes through zero linearly near the transition. A dynamical consequence is that, after the transition, the collective behavior of an ensemble of trajectories on the chaotic attractor exhibits an extreme type of intermittency. The scaling behavior of various measurable quantities near the transition is examined. [S1063-651X(96)03612-4]

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Transitions to chaos, the scenarios by which chaotic attractors arise with variation of a system parameter, are a fundamental problem in the study of nonlinear dynamical systems. So far there are four known major routes to chaotic attractors: (i) the period-doubling cascade route [1]; (ii) the intermittency transition route [2]; (iii) the crisis route [3]; and (iv) the route to chaos in quasiperiodically driven systems [4]. In the period-doubling route (i), a chaotic attractor appears in a parameter region immediately following the accumulation of an infinite number of period doublings [1]. In the intermittency route (ii), as a parameter passes through a critical value, a simple periodic orbit is replaced by a chaotic attractor in such a way that the chaotic behavior is interspersed with a periodiclike behavior in an intermittent fashion. In the crisis route (iii), a chaotic attractor is suddenly created to replace a nonattracting chaotic saddle as the parameter passes through the crisis value [3]. In systems such as the two-frequency quasiperiodically forced systems, chaos can arise through the following route (iv): (three-frequency quasiperiodicity)→(strange nonchaotic behavior)→(chaos) [4]. In the past, scaling behaviors of routes (i)–(iii) have been investigated [1,2,5]. Moreover, the dynamical picture for the route to chaos for quasiperiodically driven systems (iv) is rather clear [4,6–12], but the scaling behavior of such a transition remained unknown.

In this paper we examine the scaling with parameter of various measurable quantities near the transition to chaos in quasiperiodically driven systems. We argue both numerically through a physical model and analytically through an analyzable model, which captures the essential dynamics, that for quasiperiodically driven dynamical systems the largest nontrivial Lyapunov exponent passes through zero *linearly* with the parameter near the transition to chaos. Furthermore, near the transition, the tangent vector along a typical trajectory experiences both time intervals of expansion and time intervals of contraction. On the nonchaotic side, the Lyapunov exponent is slightly negative and, hence, contraction dominates over expansion. On the chaotic side where the Lyapunov exponent is slightly positive, expansion dominates over contraction. A striking consequence of this is that the collective behavior of an ensemble of trajectories observed at different instants of time exhibits an extreme type of intermittency on the chaotic side of the transition. During the expansion time intervals, the trajectories burst out by separating from each other, but during the contraction time intervals the trajectories merge together. Therefore, if one looks at the snapshot of slices of the attractor [13] of this ensemble of trajectories at different times, one finds that the size of the slice of the chaotic attractor varies wildly in time in an intermittent fashion. We find that the average size of the snapshot of a slice of the attractor scales *linearly* with a parameter above but near the transition. In addition, we find that the average interval between bursts also scales linearly with the parameter above the transition.

To illustrate our findings, we present results with quasiperiodic systems driven by two incommensurate frequencies. We consider the following quasiperiodically forced damped pendulum [7]:

$$\frac{d^2\theta}{dt^2}+\nu\,\frac{d\theta}{dt}+\sin\theta=K+V[\cos(\omega_1 t)+\cos(\omega_2 t)], \quad (1)$$

where θ is the angle of the pendulum with the vertical axis, ν is the dissipation rate, K is a constant, V is the forcing amplitude, and ω_1 and ω_2 are the two incommensurate frequencies. Introducing two new variables, $t\to\nu t$ and $\phi\equiv\theta+\pi/2$, Eq. (1) becomes

$$\frac{1}{p}\,\frac{d^2\phi}{dt^2}+\frac{d\phi}{dt}-\cos\phi=K+V[\cos(\omega_1 t)+\cos(\omega_2 t)],$$

where $p=\nu^2$ is a new parameter, and ω_1 and ω_2 have been rescaled accordingly: $\omega_1\to\omega_1\nu$ and $\omega_2\to\omega_2\nu$. In terms of the dynamical variables ϕ, $v\equiv d\phi/dt$, and $z\equiv\omega_2 t$, we have

$$\frac{d\phi}{dt}=v,$$

$$\frac{dv}{dt}=p\left\{K+V\left[\cos\left(\frac{\omega_1}{\omega_2}\,z\right)+\cos z\right]+\cos\phi-v\right\}, \quad (2)$$

$$\frac{dz}{dt}=\omega_2 .$$

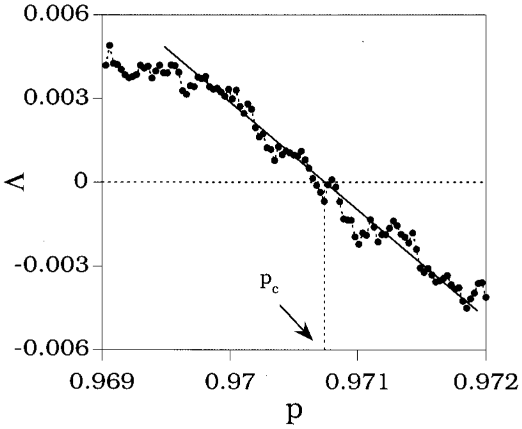

FIG. 1. For Eq. (2), the largest nontrivial Lyapunov exponent Λ versus the parameter p (damping rate) for $0.969 \leqslant p \leqslant 0.972$. Other parameter values are $V=0.55$, $K=0.8$, $\omega_2=1$, and ω_1 is chosen to be the inverse golden mean.

It has been argued that Eq. (2) exhibits rich dynamical phenomena [7,8]. In particular, for different parameters in the K-V plane, one finds two- and three-frequency quasiperiodic attractors, strange nonchaotic attractors [7,8], and chaotic attractors. Chaotic attractors can develop from two-frequency quasiperiodic attractors. It is interesting to note that in the strong damping limit $p \rightarrow \infty$, Eq. (2) reduces to a first-order equation which is isomorphic to the Schrödinger equation with quasiperiodic potentials [9].

To gain intuition, we present numerical experiments with Eq. (2) for $K=0.8$, $V=0.55$, $\omega_1=(\sqrt{5}-1)/2$ (the inverse golden mean), $\omega_2=1.0$, and choose p as the control parameter. For large values of p ($p>1.0$), the damping is strong so that the motion is typically periodic or quasiperiodic [7,8]. As p decreases, say, $p<p_c \approx 1.0$, both strange nonchaotic and chaotic attractors exist. Figure 1 shows the largest nontrivial Lyapunov exponent Λ [one Lyapunov exponent is always zero for Eq. (2)] for $p \in [0.969,0.972]$. The transition occurs at $p=p_c \approx 0.9707$ where $\Lambda>0$ for $p<p_c$ and $\Lambda<0$ for $p>p_c$. The attractors for $p<p_c$ are therefore chaotic. To visualize the attractors, we plot the variables ϕ and v on the stroboscopic surface of section defined by $z=n(2\pi)$, $n=0,1,\ldots$. Figure 2(a) shows a single long trajectory on the chaotic attractor for $p=0.9702<p_c$ ($\Lambda \approx 0.002$). Examination of the attractors for $p \gtrsim p_c$ indicates that they are strange nonchaotic [14]. One feature about the transition is that the Lyapunov exponent Λ passes through zero linearly, apart from fluctuations caused by finite length of trajectories used in numerical computation. The mechanism behind such a smooth transition can be understood by examining the relative weights of the phase-space regions where the trajectory experiences expansion and contraction [15].

To explore the properties of the chaotic attractor for $p \lesssim p_c$, we seek to study the time evolution of a snapshot of slices of the attractor [13]. Specifically, we choose a particularly relevant ensemble of initial conditions on the $[\phi,v]$ plane and let them evolve in time. These initial conditions, are chosen to have the same $z(0)=0$ (they start to evolve at the same time). A snapshot of slices of the attractor, i.e., the distribution of the trajectories resulting from these initial conditions in the phase space at fixed subsequent instants of time, are examined. We find that the properties of the snap-

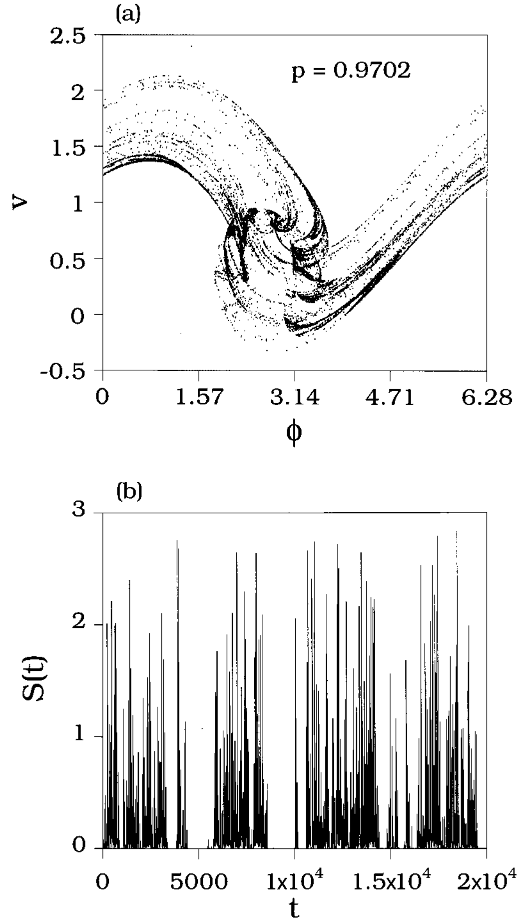

FIG. 2. (a) A single trajectory of 20 000 points on the chaotic attractor at $p=0.9702$ ($\Lambda \approx 0.002$). (b) $S(t)$, the size of the snapshot of slices of the attractor computed using 128 trajectories, versus time t for $p=0.9702$.

shot of slices of the attractor are qualitatively different for $p \gtrsim p_c$ ($\Lambda \lesssim 0$) and $p \lesssim p_c$ ($\Lambda \gtrsim 0$). In particular, for $p \gtrsim p_c$ on the nonchaotic side, the trajectories resulting from these initial conditions eventually converge to a single trajectory. At any instant of time (after sufficiently long transient time), the snapshot of a slice of the attractor of these trajectories consists of only one point in the phase space. As time progresses, the single point for all trajectories moves in the phase space, tracing out a trajectory which lies on the strange nonchaotic attractor. The time required for the ensemble of trajectories to converge to a single trajectory scales as $\tau \sim 1/|\Lambda| \sim 1/|p-p_c|$. The interesting behavior, however, occurs on the chaotic side when $p \lesssim p_c$ with Λ being slightly positive. In this case, the snapshot of slices of the attractor are no longer single points even after long transient time. There are time intervals during which the snapshot of slices of the attractor consist of points spread over the entire chaotic attractor. There are also time intervals during which the snapshot of slices of the attractor appear to consist of points concentrated on extremely small regions in the phase space. The *size* of the snapshot of slices of the attractor therefore changes drastically with time in an intermittent fashion. To quantify this situation, we define the time-dependent size of the snapshot of a slice of the attractor,

$$S(t)=\left(\frac{1}{N}\sum_{i=1}^{N}\{[\phi_i(t)-\langle\phi(t)\rangle]^2+[v_i(t)-\langle v(t)\rangle]^2\}\right)^{1/2}, \tag{3}$$

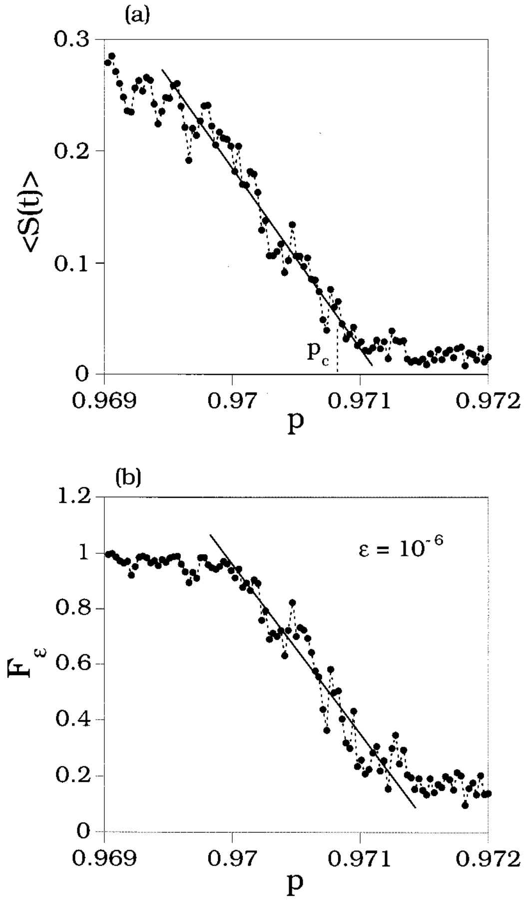

FIG. 3. (a) The average size of the snapshot of slices of the attractor versus the parameter p for $0.969 \leqslant p \leqslant 0.972$. (b) The fraction of times $F_\epsilon(p)$ at which the size of the snapshot of a slice of the attractor is larger than ϵ for $0.969 \leqslant p \leqslant 0.972$, where $\epsilon = 10^{-6}$.

where N is the number of points on the snapshot of slices of the attractor, $[\langle \phi(t) \rangle, \langle v(t) \rangle]$ defines the geometric center of these points at a given time: $\langle \phi(t) \rangle \equiv (1/N) \Sigma_{i=1}^{N} \phi_i(t)$ and $\langle v(t) \rangle \equiv (1/N) \Sigma_{i=1}^{N} v_i(t)$. Figure 2(b) shows $S(t)$ versus t for $p = 0.9702$, where t is the integer time measured on the surface of section corresponding to the real time $t(2\pi/\omega_2)$, and the snapshot of slices of the attractor are computed from 128 initial conditions uniformly chosen along the diagonal line of the rectangle defined by $\phi(0) \in [0, 2\pi]$ and $v(0) \in [-1, 1]$. It can be seen that the size of the snapshot of slices of the attractors exhibits an extreme type of intermittent behavior, the so-called ''on-off'' intermittency [16]. There are time intervals during which the snapshot of slices of the attractor are concentrated on regions with extremely small size ($< 10^{-14}$) [17]. The time averaged size of the snapshot of slices of the attractor *on the chaotic side* near the transition, defined as $\langle S(t) \rangle = \lim_{T\to\infty} (1/T) \int_0^T S(t) dt$, obeys the following scaling relation:

$$\langle S(t) \rangle \sim \Lambda \sim |p - p_c|, \tag{4}$$

as shown in Fig. 3(a), where $T = 20\,000(2\pi/\omega_2)$ with transient time $1000(2\pi/\omega_2)$ has been used in the computation. Figure 3(a) also shows that for $p \gtrsim p_c$ on the nonchaotic side, the averaged size of the snapshot of a slice of the attractor is quite small, yet nonzero. This is due to the finite transient time used in the computation. As the transient time increases, the averaged size decreases towards zero. This has been verified for several p values on the nonchaotic side. Figure 3(b) shows the fraction of times F_ϵ for which $S(t)$ is larger than ϵ versus p, where $\epsilon = 10^{-6}$. It can be seen that F_ϵ increases roughly linearly but saturates in the same range of Fig. 3(a). This indicates that the main factor contributing to the increase in $\langle S(t) \rangle$ for $p \lesssim p_c$ (the chaotic side) is a gradual filling of the ''gaps'' corresponding to the time intervals in which $S(t)$ is close to zero.

The fundamental reason for the on-off intermittent behavior in the size of the snapshot of slices of the attractor can be understood by noting that there are finite-time fluctuations in the Lyapunov exponent Λ. When Λ is slightly positive, trajectories actually experience finite-time periods when Λ is negative. Imagine that one chooses a large number of initial conditions in the phase space and compute $\Lambda(T)$ for trajectories resulting from these initial conditions over a time T. The histogram of all these $\Lambda(T)$ is usually a distribution with finite width around the asymptotic value of Λ which is only slightly positive. Thus there is a spread of the histogram into the negative side, indicating that there are trajectories experiencing contraction in finite times. This leads to the observed on-off intermittent behavior: trajectories spend stretches of time expanding (leading to nonzero-size snapshot of slices of the attractor), yet there are also long stretches of time during which the trajectories experience contraction, resulting in extremely small-size snapshot of slices of the attractor.

To understand the scaling relation Eq. (4), we note that near the transition on the chaotic side, a typical trajectory experiences slightly more expansion than contraction when it wanders on the attractor under the quasiperiodic forcing [15]. Thus we are motivated to consider here the following simple expansion-contraction model:

$$y_{n+1} = \begin{cases} 2y_n \mod(1) & \text{if } f(\theta_n) \geqslant 0 \\ y_n/2 & \text{if } f(\theta_n) < 0. \end{cases} \tag{5}$$

In the model, $f(\theta_n) = p + \cos(\theta_n)$ and $\theta_{n+1} = \theta_n + 2\pi\omega$, where ω is an irrational number in (0,1), and p is a parameter. The dynamics of θ models a quasiperiodic forcing. In contrast to the pendulum example, model (5) exhibits a transition from two-frequency quasiperiodicity to chaos. For n large, θ is uniformly distributed in $[0, 2\pi]$. Equation (5) is a modified version of the expansion-contraction model studied for random maps in Ref. [18]. Consider the case where $p \gtrsim 0$. The probability for y to expand and to contract are $P\{y_{n+1} = 2y_n\} = [2\pi - 2\cos^{-1}(p)]/2\pi \approx (\pi + 2p)/(2\pi) = 1/2 + p/\pi$, and $P\{y_{n+1} = y_n/2\} \approx 1/2 - p/\pi$, respectively. The Lyapunov exponent of the y dynamics is therefore given by $\lambda(p) = (1/2 + p/\pi)\ln 2 + (1/2 - p/\pi)\ln(1/2) = (2\ln 2/\pi)p$. Thus $\lambda(p)$ passes through zero linearly as p passes through zero, an analogous situation to Fig. 1. To compute the average size of the snapshot of slices of the attractor, we take an ensemble of initial conditions uniformly distributed in [0,1] at $n = 0$. For subsequent times, the trajectories are uniformly distributed in the interval $[0, S_n]$, where S_n can take a sequence of values $\{1, 1/2, 1/2^2, \ldots, 1/2^k\}$, where $k < n$. Thus S_n is the size of the snapshot of slices of the attractor (up to a constant proportional factor). The evolution of S_n is $S_{n+1}=1$ if $\alpha_n S_n>1$ and $S_{n+1}=\alpha_n S_n$ if $\alpha_n S_n \ll 1$, where $\alpha_n=2$ [with probability $(1/2+p/\pi)$] or 1/2 [with probability $(1/2-p/\pi)$]. For $S_n \ll 1$, letting $h_n=-\ln S_n$, we obtain a random walk in h_n: $h_{n+1}=-\ln\alpha_n+h_n$. Let $P(h,n)$ be the probability distribution function for h at time n. For $p \gtrsim 0$, the average drift of the random walk is $\nu=-\langle \ln \alpha_n\rangle=-\lambda \lesssim 0$. Thus $P(h,n)$ approximately obeys the diffusion equation

$$\frac{\partial P}{\partial n}+\nu\,\frac{\partial P}{\partial h}=D\,\frac{\partial^2 P}{\partial h^2},$$

where $D=(\ln 2-\lambda)^2(1/2+p/\pi)+(\ln 1/2-\lambda)^2(1/2-p/\pi)\approx(\ln 2)^2$ is the diffusion coefficient. When $\lambda \gtrsim 0$, the diffusion equation can be solved to yield the asymptotic distribution function $P(h)=(\lambda/D)e^{-\lambda h/D}$. The average size of the snapshot of a slice of the attractor is therefore given by $\langle S_n\rangle \approx \int_0^\infty e^{-h}P(h)dh\;=\lambda/(D+\lambda)\approx\lambda/D=[2/(\pi \ln 2)]p$ for $p\gtrsim 0$ (it can be shown that $\langle S_n\rangle=0$ asymptotically when $p<0$). We see that $\langle S_n\rangle$ increases linearly as p. It should be stressed that this linear behavior is only valid for $p\gtrsim 0$. Numerical experiments with Eq. (5) verify the linear scaling behavior of $\langle S_n\rangle$.

We remark that quasiperiodically driven dynamical systems are relevant to many physical and biological situations. The route to chaos in quasiperiodically driven systems is fundamentally different from other major routes to chaos such as the period doubling, the intermittency, and the crisis routes. The linear scaling at the onset of the on-off intermittent behavior of snapshot of slices of the attractor is a distinct physical fingerprint of the onset of chaos in quasiperiodically-driven systems.

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